Algebra PhD Qualifying Examination – August 2023

Instructions:

- Please write a neat, clear, thoughtful, and hopefully correct solution to each of the following problems. Please show all relevant work. Please clearly cross out work you do not want graded.
- You should do as many problems as the time allows. You are not expected to answer all parts of all questions in order to pass the exam.
- Each problem is worth the same. You should attempt each problem and partial credit will be given, but a complete solution of one problem is worth more than partial work on two problems.
- All rings are assumed to have 1.
- Good luck.

Problems:

1. Let G be a finite group. Say G acts on a set X. Write $g.x$ for the action of $g \in G$ on the element $x \in X$. For $x_0 \in X$, let $G.x_0 = \{g.x_0 \mid g \in G\}$ denote the orbit of x_0 and let $G_{x_0} = \{h \in G \mid h.g = g\}$ denote the stabilizer of x_0.

   (a) Prove that $G.x_0$ is a finite G-set for any fixed $x_0 \in X$.

   (b) Show that G/G_{x_0} is a G-set in a natural way. Prove that G/G_{x_0} and $G.x_0$ are isomorphic as G-sets.

   (c) If $G = C_2$ is the cyclic group of order 2 and X is a finite set with an odd number of elements, please prove that X contains a fixed point.

   (d) Fix $g_0 \in G$ and let
   $$\mathcal{C}(g_0) = \big\{h \in G \mid h = ag_0a^{-1} \text{ for some } a \in G\big\}$$
   be the set of conjugates of g_0. Prove that the order of $\mathcal{C}(g_0)$ divides the order of G.

2. (a) Please give careful statements of the three Sylow Theorems.

   (b) Please prove that any group of order $4225 = 5^2 13^2$ is abelian.

   (c) Please give a complete list of the possible groups (up to isomorphism) of order 4225.

3. Let
   $$S = \left\{a + b\sqrt[2]{3} \mid a, b \in \mathbb{Z}\right\} \subseteq \mathbb{C}.$$
   Notice that S is a subring of $\mathbb{C}$ (you don't need to check this).

   (a) Please give the definition of what it means for a commutative ring to be Noetherian.

   (b) Please prove S is a Noetherian ring.

   (c) Please prove S is a integral domain.

   (d) Please define the field of fractions of S, including the binary operations.

   (e) Please give an explicit isomorphism between the field of fractions of S and a quotient of the ring $\mathbb{Q}[x]$. Of course, you should prove that the map you give is an isomorphism.

4. Let R be a commutative ring. If M is an R-module, then write

$$\mathrm{Ann}_R(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$$

for the annihilator of M.

(a) Please prove that $\mathrm{Ann}_R(M)$ is an ideal of R.

(b) Please prove that if $J \trianglelefteq R$ is an ideal, then $J = \mathrm{Ann}_R(M)$ for some R-module M.

(c) Give an example of a commutative ring R and an R-module M with the property that for each $m \in M$ we have

$$\{r \in R \mid rm = 0\} \neq \{0\}\,,$$

but $\mathrm{Ann}_R(M) = \{0\}$. Of course, you should explain why your example satisfies the given requirements.

5. Let R be a commutative ring. If M, N, and P are R-modules and $f : M \to N$ and $g : N \to P$ are R-module homomorphisms, then we say

$$M \xrightarrow{f} N \xrightarrow{g} P$$

is *exact at* N if $\mathrm{Im}(f) = \mathrm{Ker}(g)$.

(a) If N and P are R-modules and $0 = \{0\}$ is the zero R-module, then please prove $N \xrightarrow{g} P \to 0$ is exact at P if and only if g is surjective.

(b) Assume

$$0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$$

is exact at M, N, and P. Assume there is an R-module homomorphism $\varphi : P \to N$ such that $g \circ \varphi = \mathrm{Id}_P$. Please give an explicit R-module isomorphism between N and $M \oplus P$. Of course, you should explain why your map is an isomorphism.

6. Let $f(x) = x^3 - 5 \in \mathbb{Q}[x]$. Let $K \subseteq \mathbb{C}$ be the splitting field for $f(x)$ thought of as a field extension of $\mathbb{Q}$.

(a) Please compute $[K : \mathbb{Q}]$ and give a basis for K as a $\mathbb{Q}$-vector space.

(b) Please compute the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$.

(c) Please determine the intermediate fields between $\mathbb{Q}$ and K along with their lattice of containments.