TOPOLOGY QUALIFYING EXAM — AUGUST 2022

1. Definitions and examples

Please clearly state definitions, and describe your examples precisely. You do NOT need to prove that your examples have the required properties. **Solve *all* problems in this section.**

Problem 1.1. Define what it means for a topological space to be *compact*. Define what it means for a topological space to be *Hausdorff*. A theorem from class states that a compact subspace of a Hausdorff space is closed. Give an example which shows that the Hausdorff assumption in this theorem is necessary.

Problem 1.2. Give the definition of the *one-point compactification* topology on $X \cup \{\infty\}$ where X is a non-compact, locally compact, Hausdorff topological space. Give an example of non-homeomorphic, non-compact, locally compact, Hausdorff spaces X and Y whose one-point compactifications are homeomorphic.

Problem 1.3. Let $(X, \mathcal{T})$ be a topological space, $\sim$ an equivalence relation on the set X, and let $X/{\sim}$ denote the set of all equivalence classes. Define the *quotient topology* on the set $X/{\sim}$. What is the quotient space $\mathbb{R}^2/{\sim}$ where $(x_1, y_1) \sim (x_2, y_2)$ iff $y_1 - x_1^3 = y_2 - x_2^3$? (It is a well known space).

Problem 1.4. Given a continuous map $f : X \to Y$ between topological spaces, and $x_0 \in X$ a basepoint, define the *induced homomorphism* f_* between the appropriate fundamental groups. Give an example where f is injective, but f_* is not injective. Give an example where f is surjective, but f_* is not surjective.

Problem 1.5. Give the definition of a *covering space map* $p : E \to B$ between topological spaces. Give the definition of a *local homeomorphism* $f : X \to Y$. Give an example of topological spaces X and Y and a surjective map $p : X \to Y$ which is a local homeomorphism but which is not a covering space map.

2. Point-set topology

Solve 3 of the following problems. Please clearly indicate what theorems you are using.

(The set $\mathbb{R}$ of real numbers is always endowed with the standard topology, $\mathbb{R}^n$ with the product topology, and subsets like $[0,1] \subset \mathbb{R}$, $S^1 \subset \mathbb{R}^2$, etc with the subspace topology.)

Problem 2.1 (Basis)**.** (a) Give the definition of a basis $\mathcal{B}$ for a topology on a set X.

(b) Let $(X, \mathcal{T})$ be a topological space. How do you recognize a basis for the topology $\mathcal{T}$? That is, state a theorem which guarantees that a collection of open sets $\mathcal{C} \subset \mathcal{T}$ is a basis for the given topology $\mathcal{T}$.

(c) Give an example of a topological space $(X, \mathcal{T})$ and a subcollection $\mathcal{C} \subseteq \mathcal{T}$ which satisfies the conditions in part (a) above but for which the induced topology on X is distinct from $\mathcal{T}$.

Problem 2.2 (Compactness)**.** (a) Prove that a continuous bijection from a compact to a Hausdorff space is a homeomorphism. State clearly (without proofs) the basic results about compact spaces that you use in your proof.

(b) Give an example of a continuous bijection between topological spaces which is not a homeomorphism.

(c) Let $\mathbb{R}^2$ have the usual topology and let $p_1 : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x$ and $p_2 : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto y$ be the projection maps. Suppose $C \subset \mathbb{R}^2$ is a subspace such that $p_1(C)$ and $p_2(C)$ are closed and bounded in $\mathbb{R}$. Is C compact? Prove or give a counterexample.

Problem 2.3 (Metric Spaces)**.** Suppose that (X, d) is a complete metric space and that D_n is a sequence of closed subspaces of X satisfying the following two conditions for all $n \in \mathbb{Z}_+$.

- $D_{n+1} \subseteq D_n$
- there exists $x_n \in D_n$ such that $D_n \subseteq B_{\frac{1}{n}}(x_n)$, the open d-metric ball of radius $\frac{1}{n}$ about x_n.

Prove the following.

(a) The sequence (x_n) is Cauchy.

(b) The sequence (x_n) has a limit $x \in X$.

(c) $x \in D_n$ for all $n \in \mathbb{Z}_+$.

Problem 2.4 (Connectedness)**.** (a) Let X be a topological space, and $U \subset X$ a subset. Prove directly from the definition that, if U is connected, then its closure $\bar{U}$ is also connected.

(b) Suppose that $U \subset X$ is a connected subspace of X as above. Is the interior of U also connected? Prove or give a counterexample.

(c) Write down an example of a connected space which is not path connected (no proof needed).

Problem 2.5 (Quotient spaces)**.** Let $\sim$ denote the equivalence relation on $\mathbb{R}^2$ (using column vector notation) given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \Longleftrightarrow \quad \begin{pmatrix} x - x' \\ y - y' \end{pmatrix} \in \mathbb{Z}^2$$

and let

$$q : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \left[\begin{pmatrix} x \\ y \end{pmatrix} \right]$$

be the corresponding quotient map. Let A be a (2×2)-matrix with integer entries and determinant equal to 1. Carefully explain how A induces a homeomorphism

$$\widehat{A} : \mathbb{R}^2/\mathbb{Z}^2 \;\to\; \mathbb{R}^2/\mathbb{Z}^2.$$

3. Fundamental group and covering spaces

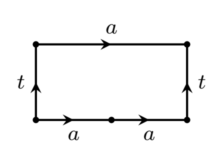
Solve 3 of the following problems. Please clearly indicate what theorems you are using.

(The set $\mathbb{R}$ of real numbers is always endowed with the standard topology, $\mathbb{R}^n$ with the product topology, and subsets like $[0, 1] \subset \mathbb{R}$, $S^1, D^2 \subset \mathbb{R}^2$, etc with the subspace topology.)

Problem 3.1 (Retractions)**.** (a) Suppose that $j : A \hookrightarrow X$ is the inclusion map of the subspace $A \subseteq X$ and that A is a retract of X. Prove that, for $x_0 \in A$, the inclusion induced map $j_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is an injection.

(b) Prove that there is no retraction from the solid torus $D^2 \times S^1$ to its boundary $S^1 \times S^1$. Here S^1 is the unit circle and D^2 the unit disk in the plane $\mathbb{R}^2$.

Problem 3.2 (Cell Complexes; van Kampen)**.** Consider a cell complex X with 1-skeleton $S^1_a \vee S^1_t$ (one 0-cell v and two oriented 1-cells labeled a and t) and 2-cell shown.



(a) Write down a presentation for the fundamental group $\pi_1(X, v)$. (No need for any detailed justifications)

(b) Compute the abelianization of the group $\pi_1(X, v)$ obtained above.

(c) Is the universal covering space of X a compact space?

Problem 3.3 (Covering spaces and subgroups)**.** (a) Prove that if $p : (E, e_0) \to (B, b_0)$ is a covering space map, then p_* is injective. State any theorem that you use in your proof.

(b) Prove that there is no covering space map $T^2 \to \mathbb{R}\mathrm{P}^2$ where T^2 is the 2-torus and $\mathbb{R}\mathrm{P}^2$ is the real projective plane.

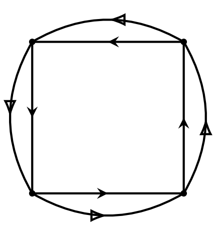
(c) Prove that there is no covering space map $\mathbb{R}\mathrm{P}^2 \to T^2$ where T^2 and $\mathbb{R}\mathrm{P}^2$ are defined above.

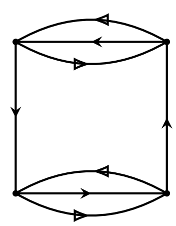
Problem 3.4 (General Lifting Theorem)**.** (a) State the general lifting theorem for continuous maps $f : (Y, y_0) \to (B, b_0)$ into the base space of a covering space $p : (E, e_0) \to (B, b_0)$. Note that this theorem involves topological conditions on the space Y as well as an algebra condition.

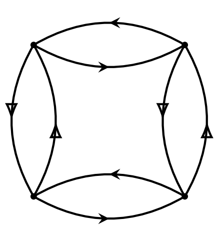
(b) Prove that every continuous map $f : \mathbb{R}\mathrm{P}^2 \to T^2$ is null-homotopic. Here $\mathbb{R}\mathrm{P}^2$ is the real projective plane and T^2 is the 2-torus.

Problem 3.5 (Deck Transformations)**.** (a) Let $p : (E, e_0) \to (B, b_0)$ be a path connected and locally path connected covering space. Write down an algebraic expression (no need to justify the expression) for the deck transformation group $\mathrm{Aut}(E \to B)$ in terms of the fundamental groups of E and B.

(b) Let $B = S^1 \vee S^1$ be the wedge of two circles. Determine the group $\mathrm{Aut}(E \to B)$ for each of the following 4 covering spaces.

I.


II.


III.


IV.