Real Analysis
Fall, 2022

Qualifying Exam

Jing Tao
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General Instructions: You have three hours and you may use as much paper as you need. Please try to write neatly and concisely, justify every step and state all theorems that you are using. Prominently label the problem number next to each solution.

1

Let $A \subset [0,1]$ be the subset of real numbers whose decimal expansions contain no 3's. Prove that A is Lebesgue measurable, and find its measure. (Some real numbers have non-unique decimal expansions, but note that this does not cause any issues here.)

2

Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose a sequence of measurable functions $\{f_n\}$ on X converges in measure to a measurable function f. Assume $0 \leq f_n(x) \leq 1$ for μ-a.e. $x \in X$.

(a) Prove that $0 \leq f(x) \leq 1$ for μ-a.e. $x \in X$.

(b) Prove that $\int_X f \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu$.

(c) Give an example where the inequality in part (b) is strict.

3

Compute the following limit justifying all steps

$$\lim_{n\to\infty} \int_0^\infty \frac{n \sin\left(\frac{x}{n}\right)}{x(1+x^2)} \, dx$$

4

Let m be the Lebesgue measure on $\mathbb{R}$. Let $A \subset [0,1]$ be a Lebesgue measurable set such that there exists a constant $\alpha > 0$ such that for every interval $I \subset [0,1]$ one has $m(A \cap I) \geq \alpha \, m(I)$. Prove that A has full measure, i.e. $m(A) = 1$.

5

Let $f, g \in L^1(\mathbb{R})$. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by

$$F(x, y) = f(x - y)g(y).$$

(a) Explain why F is measurable.

(b) Prove that $F \in L^1(\mathbb{R}^2)$.

6

Let $f \in L^p(X, \mathcal{M}, \mu)$ for some $1 \leq p < \infty$. Define $g : [0, \infty) \to [0, \infty]$ by

$$g(t) = \mu\{x \in X \mid t \leq |f(x)|\}$$

Observe that g is monotonic, hence Borel measurable. Prove that

$$||f||_p = \left(\int_0^\infty g(t^{1/p})\, dt \right)^{1/p}.$$

Hint: you may want to consider when f is a simple function first.

7

Let $p \in [1, \infty)$. Show that for all $\lambda < 1 - \frac{1}{p}$, then

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon^\lambda} \int_0^\epsilon f(x)\, dx = 0 \quad \forall f \in L^p([0,1]).$$

8

Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that

(a) $f \in L^p(\mathbb{R})$ if and only if $1 < p < 2$.

(b) $f \in L^p(\mathbb{R})$ if and only if $p \geq 2$. Hint: Use $f(x) = (\ln x)^a x^b \chi_{[1,\infty)}$ for appropriate a and b.