ALGEBRA QUALIFYING EXAM - AUGUST 2022

Instructions

Solve all the problems. Give clear and complete justification of your solutions, unless prompted otherwise for a particular problem.

Problems

1. Determine if each of the following rings is a unique factorization domain. For each case, you need only give a brief justification.
   (a) $\mathbb{Z}[2\sqrt{2}]$
   (b) $\mathbb{Z}[x, y]$
   (c) $\mathbb{Z} + x\mathbb{Q}[x] = \{a_0 + a_1x + \cdots + a_nx^n \mid a_0 \in \mathbb{Z}, a_1, \ldots, a_n \in \mathbb{Q}, n \in \mathbb{Z}^+\}$. (Hint: consider the element x).
2. Let R be a ring (associative with 1) with finitely many elements. Prove that if R cannot be written as a direct product $R = R_1 \times R_2$ of smaller rings, then the number of elements of R is a power of a prime.
3. Prove that the additive group $\mathbb{Q}$ of rational numbers is not a free $\mathbb{Z}$-module.
4. Put the matrix $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \in M_3(\mathbb{Z})$ in Rational Canonical Form.
5. Suppose that A is a 7×7 complex matrix such that $A^5 = 2A^4 + A^3$. Further, say that rk $A = 5$ and tr $A = 4$, where rk indicates the rank and tr indicates the trace of a matrix. Find the Jordan canonical form of A.
6. Consider $\sqrt{5} - \sqrt{2}$. Determine the degree of the field extension of $\mathbb{Q}$ by $\sqrt{5} - \sqrt{2}$ and find its minimal polynomial.
7. (a) Let α be a root of the polynomial $x^4 - 7 \in \mathbb{Q}[x]$. Show that $\mathbb{Q}(\alpha)$ is not Galois over $\mathbb{Q}$.
   (b) Let F be a field of order 5 and let β be a root of $x^4 - 7 \in F[x]$. Show that $F(\beta)$ is Galois over F.
   (c) Compute the Galois group from part (b); that is, compute $Gal(F(\beta)/F)$. Further, tell us what familiar group it is isomorphic to.
8. Classify up to isomorphism all groups G of order 56 with the property that all Sylow subgroups of G are cyclic. Write down a presentation for each group you find.