Algebra Qualifying Exam — January 2022

Instructions

- Do as many problems as you are able to.
- Give full and clear justification of your solutions.
- Throughout, $\mathbb{F}_p$ denotes a field with p elements, where p is a prime number.
- For a ring R, $M_n(R)$ denotes the ring of $n \times n$ matrices over R.

Problems

1. (a) State and prove the class equation for a finite group G. (This is the formula relating $|G|$ to the sizes of centralizers.)

   (b) Use (a) to prove that any nontrivial finite p-group has nontrivial center.

2. (a) True or false (prove your assertion): If p is a prime element of a commutative integral domain R, then p is irreducible.

   (b) For $n \in \mathbb{Z}$, determine the set of prime ideals of $R = \mathbb{Z}/n\mathbb{Z}$.

   (c) True or false (prove your assertion): Every prime ideal of $\mathbb{Q}[x, y]$ is principal.

3. (a) Let R be a commutative ring, M and N be R-modules, and $\mathrm{Hom}_R(M, N)$ be the set of R-module homomorphisms from M to N. Describe a natural R-module structure on $\mathrm{Hom}_R(M, N)$.

   (b) With notation as in (a), describe also a natural $\mathrm{End}_R(M)$-module structure on $\mathrm{Hom}_R(M, N)$.

   (c) Let M be the $\mathbb{Z}$-module $\mathbb{Z} \oplus 2\mathbb{Z}$, N be the $\mathbb{Z}$-module $\mathbb{Z} \oplus 3\mathbb{Z}$. Determine the structure of $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ as a $\mathbb{Z}$-module.

4. Prove that, for any integer $n \geq 2$, there is a monic polynomial $f(x) \in \mathbb{Q}[x]$ of degree n with no rational or repeated roots, such that $f(x) = 0$ is solvable by radicals.

5. (a) Determine the number of irreducible quadratic polynomials over $\mathbb{F}_p$.

   (b) Determine the number of quadratic field extensions $E/\mathbb{F}_p$ (up to isomorphism).

   (c) Let $E/\mathbb{F}_p$ be a *quadratic skew-field extension of* $\mathbb{F}_p$, i.e., a division ring of order p^2. Prove that $\mathbb{F}_p$ embeds in E (as rings) and E is a 2-dimensional vector space over $\mathbb{F}_p$ (justifying the term "extension").

   (d) Determine the number of quadratic skew-field extensions $E/\mathbb{F}_p$ (up to isomorphism).

6. Let S be the set of all irreducible monic polynomials $f(x) \in \mathbb{Q}[x]$ whose splitting field is isomorphic to $\mathbb{Q}(i, \sqrt{-3})$.

   (a) Determine $\{\deg(f) : f \in S\}$.

   (b) Construct some $f \in S$.

7. Consider the map $\iota(x) = (\operatorname{tr} x)I - x$ on $M_2(\mathbb{Q})$.

(a) Prove that ι is an involution of $M_2(\mathbb{Q})$, i.e., an additive map of order 2 which reverses the order of multiplication.

(b) Show $x \cdot \iota(x) = (\det x)I$ for $x \in M_2(\mathbb{Q})$.

(c) Suppose F is a subring of $M_2(\mathbb{Q})$ which is a field different from $\mathbb{Q}$. Prove that ι restricts to a nontrivial Galois automorphism of F.