TOPOLOGY QUALIFYING EXAM — AUGUST 2021

1. Definitions and examples

Please clearly state definitions, and describe your examples precisely. You do *not* need to prove that your examples have the required properties. **Solve *all* problems in this section.**

Problem 1.1. Let $(X, \mathcal{T})$ be a topological space. Define basis for the topology $\mathcal{T}$. Give an example of a topological space $(X, \mathcal{T})$ and two *different* bases $\mathcal{B}, \mathcal{B}'$ for the same topology $\mathcal{T}$.

Problem 1.2. Define what it means for a metric space (X, d) to be complete. Give an example of a complete metric space.

Problem 1.3. Define covering map $p : X \to Y$.

Problem 1.4. Let X, Y topological spaces and $f, g : X \to Y$ continuous functions. Define what it means for f to be homotopic to g. In case $X = [0, 1]$, define when the paths f, g are *path*-homotopic.

Problem 1.5. Define what it means for a continuous function $f : X \to Y$ to be a quotient map. Give two concrete examples of continuous surjections for which one is a quotient map and the other is *not*.

2. Point-set topology

Solve 3 of the following problems. Please clearly indicate what theorems you are using.

(The set $\mathbb{R}$ of real numbers is always endowed with the standard topology, $\mathbb{R}^n$ with the product topology, and subsets like $\mathbb{Z}, \mathbb{Q} \subset \mathbb{R}$, $S^1 \subset \mathbb{R}^2$, etc with the subspace topology.)

Problem 2.1. Let $X = \mathbb{Q}$, and $Y = \mathbb{Z}$. Prove that the connected components of X and Y are just points. Prove that X and Y are *not* homeomorphic.

Problem 2.2. Let $X = \mathbb{R}^2/\sim$ be the quotient topological space, where the equivalence relation is given by

$$(x, y) \sim (z, w) \iff x^2 + y^2 = z^2 + w^2.$$

Prove that X is homeomorphic to $[0, \infty)$.

Problem 2.3. Let X be a metric space, and assume $Y \subset X$ is a countable dense subset. Prove that X is second countable.

Problem 2.4. Let $Y \subset \mathbb{R}^2$ be given by $Y = \mathbb{R} \times \{1, 2\}$. Let X be the quotient space $X = Y/\sim$, where the equivalence relation is given by $(x, 1) \sim (x, 2)$ for all $x \neq 0$. (In words, X is obtained from two disjoint copies of the real line by identifying corresponding points *except for the origins*.)

(a) Prove that X is *not* Hausdorff.
(b) Prove that X is path-connected.

Problem 2.5. (a) Prove that if $f : X \to Y$ is a continuous map, and X is compact, then the image $f(X)$ is compact.
(b) Prove that, if X is Haudorff, and $Y \subset X$ is compact, then Y is a closed subset of X.
(c) Prove that, if X is compact, Y is Hausdorff, and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

3. Fundamental group and covering spaces

Solve 3 of the following problems. Please clearly indicate what theorems you are using.

(The set $\mathbb{R}$ of real numbers is always endowed with the standard topology, $\mathbb{R}^n$ with the product topology, and subsets like $\mathbb{Z}, \mathbb{Q} \subset \mathbb{R}$, $S^1 \subset \mathbb{R}^2$, etc with the subspace topology.)

Problem 3.1. Let X be a path-connected and locally path-connected topological space with $\pi_1(X)$ finite, and let $T^2 = S^1 \times S^1$ denote the 2-dimensional torus.

(a) Prove that every continuous map $f : X \to T^2$ is homotopic to a constant map. (Hint: use covering spaces.)
(b) Prove there is no covering space map $T^2 \to X$.

Problem 3.2. Prove there is no retraction $S^1 \times D^2 \to S^1 \times S^1$, where $D^2 \subset \mathbb{R}^2$ denotes the closed disk with radius one centered at the origin, and $S^1 \subset D^2$ is its boundary circle.

Problem 3.3. If x, y belong to the same path-component of a space X, prove that $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic.

Problem 3.4. (a) Prove that the groups $\mathbb{Z}_2 * \mathbb{Z}_3$ and $\mathbb{Z}_3 * \mathbb{Z}_3$ are not isomorphic.
(b) Construct two path-connected topological spaces with fundamental groups isomorphic to these two groups.

Problem 3.5. Describe *all* path-connected 3-fold covers of $X = S^1 \vee \mathbb{R}P^2$ (please justify why your list is exhaustive). Which are regular (i.e., normal) and why?