Qualifying exam Analysis January 2021

1. **(3+3 points)** Find the σ-algebras $\mathcal{M}_j = \sigma(\mathcal{C}_j)$ on $\mathbb{R}$ that are generated by the following collections of sets:

$$\mathcal{C}_1 = \{\{x\} : x \in \mathbb{R}\}, \quad \mathcal{C}_2 = \{(n, n+x) : n \in \mathbb{Z}, x > 0\}$$

2. **(1+4+1 points)** Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f_n : X \to [0, \infty]$ be a sequence of measurable functions.

   (a) State the *monotone convergence theorem* (in this setting).

   (b) Now suppose also that $f(x) := \lim f_n(x) \in [0, \infty]$ exists for almost every $x \in X$, and $f_n(x) \leq f(x)$ almost everywhere for all $n \geq 1$. Show that then
$$\int_X f(x)\, d\mu(x) = \lim_{n\to\infty} \int_X f_n(x)\, d\mu(x).$$
   Suggestion: Use Fatou's lemma.

   (c) Can you also use *dominated convergence* to prove the result from part (b)?

3. **(5 points)** Let μ, ν be finite measures on a common space $(X, \mathcal{M})$. Show that there is a measurable function $f : X \to [0, 1]$ such that
$$\int_A (1-f)\, d\mu = \int_A f\, d\nu$$
   for all $A \in \mathcal{M}$. *Suggestion:* Use the Radon-Nikodym theorem.

4. **(2+2+2+2 points)** Consider the sequence of functions $f_n \in L^1(\mathbb{R})$, $f_n(x) = \chi_{(n,2n)}(x)$. Does f_n converge:
   (a) pointwise almost everywhere (with respect to Lebesgue measure);
   (b) in L^1;
   (c) in measure;
   (d) in $\mathcal{D}'(\mathbb{R})$?
   In those cases where the sequence does converge, please also identify the limit.

5. **(6 points)** Evaluate

$$\int_0^1 dx \int_0^{\sqrt{\pi}} dy\, y^3 \cos(xy^2)$$

You will probably want to use Fubini-Tonelli here. Please justify this carefully; don't just do the formal calculation.

6. **(1+2+3 points)** Find all p, $1 \leq p \leq \infty$, for which $f \in L^p(\mathbb{R})$, for the following functions:

$$\text{(a) } f(x) = 1; \quad \text{(b) } f(x) = xe^{-|x|};$$
$$\text{(c) } f(x) = \sum_{n=4}^{\infty} n^{-1/2}(x-n)^{-1/n}\chi_{(n,n+1)}(x)$$

7. **(4 points)** Let $f : (0,1) \to (0,\infty)$ be a measurable function. Prove that

$$\int_0^1 f(x)\, dx \int_0^1 \frac{1}{f(x)}\, dx \geq 1.$$

Suggestion: Try to apply Hölder's inequality.

8. **(4+2 points)** (a) Let $u \in \mathcal{S}'(\mathbb{R})$ be a tempered distribution with $(d^N/dx^N)u = 0$ for some $N \geq 0$. Show that then the Fourier transform $\widehat{u} \in \mathcal{S}'(\mathbb{R})$ satisfies $(\widehat{u}, \varphi) = 0$ for all $\varphi \in C_0^\infty(\mathbb{R})$ with $0 \notin \operatorname{supp} \varphi$.

   (b) Find $\widehat{u} \in \mathcal{S}'(\mathbb{R})$ for $u(x) = x$ (that is, $u \in \mathcal{S}'(\mathbb{R})$ is the distribution $u = u_f$ that is generated by the function $f(x) = x$).