Topology Qualifying Exam
January 10, 2020

Part I, Definitions/Examples: *Work all problems in this part. Thoroughly state each definition and fully describe any examples, but it is not necessary to give complete proofs of your assertions.*

1. Let X be a space and let A be a subset of X. Define the closure $\overline{A}$ of A. Show that if $\{A_\alpha \mid \alpha \in J\}$ is a family of subsets of X then $\bigcup_{\alpha\in J} \overline{A_j}$ need not equal $\overline{\bigcup_{\alpha\in J} A_j}$.

2. Let X and Y be spaces. Define what it means for two continuous functions f and g from X to Y to be homotopic. Define what it means for two paths in X to be path homotopic.

3. Let (X, x_0) and (Y, y_0) be spaces and let $f : X \to Y$ be a continuous function. Describe the homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. Show that even if f is surjective, f_* need not be surjective.

4. Define the product topology on $\prod_{\alpha\in J} X_\alpha$ where each X_α is a space with topology $\mathcal{T}_\alpha$. Define the projection map from $\prod_{\alpha\in J} X_\alpha$ to X_{α_0} for a given element $\alpha_0 \in J$, and show that this function may not be a closed map.

5. Let X be a space and let $f : X \to Y$ be a surjection. Define the quotient topology on Y induced by f. Give an example of a continuous function $f : X \to Y$ where the topology on Y is not the quotient topology.

Part II: *Work three problems from this part. Unless indicated otherwise, proofs should be thorough and rely only on basic definitions and major theorems. Any theorems that are invoked should be carefully stated, and definitions should be either explicitly given or made clear from the context of the explanation.*

6. Show that the product of two spaces that are regular and second countable is also regular and second countable.

7. Suppose that X is a space which is the countable union of connected subspaces A_n for all integers $n > 0$ and assume that $A_n \cap A_{n+1} \neq \emptyset$ for each n. Show that X is connected.

8. (a) Show that the collection of intervals $\{[a, b) \mid a < b\}$ is a basis for a topology $\mathbb{R}_\ell$ on $\mathbb{R}$.
(b) Show that the subspace topology on a non-vertical line in $\mathbb{R}_\ell \times \mathbb{R}_{euclid}$ is homeomorphic to $\mathbb{R}_\ell$.
(c) Show that if there is a continuous path between two points in $\mathbb{R}_\ell \times \mathbb{R}_{euclid}$ then the points lie on a vertical line.

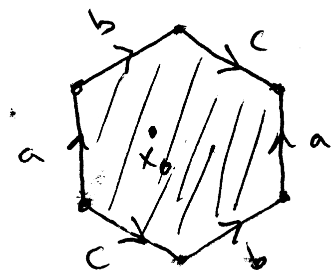
9. (a) Show that the image of a compact space under a continuous function is compact.
(b) Show that a compact subset of a Hausdorff space is closed.
(c) Show that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

10. Let (X, d) be a metric space and $A \subseteq X$. Show that $x \in X$ is a limit point of A if and only if there is a sequence in A whose limit is x.

Part III: *Work three of the problems from this part using the same guidelines as in Part II.*

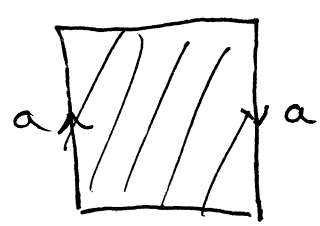
11. Let X be a space with basepoint x_0.
(a) Thoroughly define the elements $[\alpha]$ of $\pi_1(X, x_0)$ and the group operation.
(b) Describe the inverse $[\alpha]^{-1}$ of $[\alpha] \in \pi_1(X, x_0)$, and verify that it satisfies the inverse laws $[\alpha]^{-1}[\alpha] = 1 = [\alpha][\alpha]^{-1}$ by describing any requisite homotopies.

12. Let X be the 2-dimensional space obtained from a hexagon by identifying opposite faces as indicated in the picture.
(a) Outline how to derive a presentation for the fundamental group $\pi_1(X, x_0)$ using Van Kampen's theorem.
(b) Is the group $\pi_1(X, x_0)$ abelian? Justify your answer.
(c) Show that there is a regular covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ with degree n for every integer $n > 0$.



13. Construct a compact space X which is path connected and locally simply connected and whose fundamental group is isomorphic to the free product $\mathbb{Z}_4 * \mathbb{Z}_5$ of cyclic groups with order 4 and 5. Show that the universal cover of X cannot be compact.

14. The Mobius band M is the quotient space obtained from $[0, 1] \times [0, 1]$ by identifying $(0, t)$ with $(1, 1 - t)$ as shown in the picture below.
(a) Show that the subset of M represented by all points $(x, 1/2)$ (with $(0, 1/2)$ and $(1, 1/2)$ identified) is a deformation retract of M.
(b) Show that the boundary of M (which is represented by all points (x, t) where $t = 0$ or $t = 1$) is not a retract of M.



15. Let S_g denote the closed orientable surface of genus g. Suppose that $F : (S_3, x_0) \to (S_2, y_0)$ is a continuous function whose induced homomorphism F_* is one-to-one. Show that the image of F_* has index two in $\pi_1(S_2, y_0)$.