Algebra Qualifying Exam — January 2020

Instructions:

- Please write a neat, clear, thoughtful, and hopefully correct solution to each of the following problems. Please show all relevant work.
- You should do as many problems as the time allows. You are not expected to answer all parts of all questions in order to pass the exam. In particular, even if you haven't solved the earlier parts of a problem, you are free to assume they are true and use them to solve the later parts.
- Each problem is worth the same. Partial credit will be given, but a complete solution of one problem is worth more than partial work on two problems.
- Good luck.

$\boxed{1}$ Let G be a group and let $A, B \trianglelefteq G$ be normal subgroups of G.

(a) Show that $A \cap B$ is also a normal subgroup of G.
(b) Prove that, if both A and B have finite index, so does $A \cap B$.
(c) Consider the function $\phi : G \to G/A \times G/B$ given by $\phi(g) = (gA, gB)$. Prove that ϕ is a homomorphism and compute its kernel.
(d) Prove that, if the orders of G/A and G/B are finite and coprime, then there is an isomorphism $G/(A \cap B) \simeq G/A \times G/B$.

$\boxed{2}$ (a) Give the definition of a Principal Ideal Domain (PID).
(b) Show that the polynomial ring $\mathbb{Z}[x]$ is not a PID.
(c) Find all units in the ring of Gaussian integers $\mathbb{Z}[i]$.

$\boxed{3}$ Let G be a group of order $|G| = 125$ acting on a set X with cardinality $|X| = 100$.

(a) What are the possible cardinalities of an orbit of this action?
(b) Prove that, if the action has at least one fixed point, then it must have at least 5 fixed points.
(c) What is the fewest number of orbits that an action of G on X can have? Make sure to prove that such number of orbits is possible by giving an example.

$\boxed{4}$ Let K/F be a degree 15 Galois extension. List all subfields of K which contain F, as well as their degree over F.

$\boxed{5}$ Let $f : R \to S$ be a unital ring homomorphism between two commutative unital rings. Prove or disprove: If $I \subset S$ is a prime ideal, then the preimage $f^{-1}(I) \subset R$ is a prime ideal.

$\boxed{6}$ Let R be a commutative ring, and let I and J be two ideals in R. Consider two R-module maps:

- $f : I \cap J \to I \oplus J$, where $f(x) = (x, -x)$.
- $g : I \oplus J \to I + J$, where $g(x, y) = x + y$.

Prove that the following sequence of R-modules is exact.

$$0 \longrightarrow I \cap J \xrightarrow{f} I \oplus J \xrightarrow{g} I + J \longrightarrow 0$$

$\boxed{7}$ Let F be a field whose characteristic is not 23, and let α, β be distinct elements in F such that

$$\alpha^3 = \alpha + 1 \text{ and } \beta^3 = \beta + 1$$

Prove that F contains a third element γ such that $\gamma^3 = \gamma + 1$. Make sure to prove that your γ is distinct from α and β.