Algebra Qualifying Exam — August 2019

Instructions:

- Please write a neat, clear, thoughtful, and hopefully correct solution to each of the following problems. Please show all relevant work.
- You should do as many problems as the time allows. You are not expected to answer all parts of all questions in order to pass the exam. In particular, even if you haven't solved the earlier parts of a problem, you are free to assume they are true and use them to solve the later parts.
- Each problem is worth the same. Partial credit will be given, but a complete solution of one problem is worth more than partial work on two problems.
- Good luck.

$\boxed{1}$ (a) Let G be a group and consider an element $g \in G$. Give the definition of the *centralizer* $C_G(g)$ of g in G.

(b) Let G be a finite group and let $g_1, g_2, \ldots, g_n \in G$ be representatives of the conjugacy classes of G. Show that

$$1 = \frac{1}{|C_G(g_1)|} + \frac{1}{|C_G(g_2)|} + \cdots + \frac{1}{|C_G(g_n)|}.$$

(c) Classify (up to isomorphism) all finite groups that have exactly 3 conjugacy classes.

$\boxed{2}$ (a) State the Fundamental Theorem of Finitely Generated Abelian Groups.

(b) Show that a group of order 45 cannot be simple.

(c) Classify (up to isomorphism) all groups of order 45.

$\boxed{3}$ Let R be a subring of a field F such that for every $x \in F$ either $x \in R$ or $x^{-1} \in R$ (or both). Prove that if I and J are two ideals in R, then either $I \subseteq J$ or $J \subseteq I$.

$\boxed{4}$ Let A and B be a pair of $n \times n$ matrices over $\mathbb{R}$ that commute; that is, $\mathsf{AB} = \mathsf{BA}$. Let $I \subset \mathbb{R}[x, y]$ be the set of *relations* between A and B; that is, polynomials $f(x, y)$ such that $f(\mathsf{A}, \mathsf{B})$ is the zero matrix. Prove that I is a finitely generated ideal in $\mathbb{R}[x, y]$.

$\boxed{5}$ Let R be a Noetherian ring, and let $f : R \to R$ be a surjective ring homomorphism. Prove that f is an isomorphism.

$\boxed{6}$ Let a, b be unknown integers, and let γ denote the real number $\sqrt{a} + \sqrt{b}$.

(a) Show that $\mathbb{Q}(\gamma)$ contains $\sqrt{a}$.

(b) Determine the degree of the extension $\mathbb{Q}(\gamma)$ over $\mathbb{Q}$. Your answer may split into cases based on the values of a and b.

(c) Determine whether the extension $\mathbb{Q}(\gamma)$ over $\mathbb{Q}$ is Galois. Your answer may split into cases based on the values of a and b.
