Qualifying Exam in Topology, Fall 2018

1. Definitions and Theorems

Define the following terms/state the following theorems. Definitions/Theorems must be stated in full.

1. Basis for a topology
2. Tychonoff's Theorem
3. $\pi_1(X, x)$, i.e. the fundamental group of a topological space X with basepoint x. (The definition should include the definition of the group operation. Just state definitions, no verification of well-definedness required.)
4. Connected component
5. Van Kampen's Theorem

2. Point Set Topology (Mostly)

Solve 4 of the following problems. In your answers, indicate what theorems that you are using.

(1) Suppose $f : S^2 \to S^2$ is continuous. Show that f is a closed map.

(2) For each of the following, prove it's true or prove it's false with a counterexample.

- (a) If X and Y are homeomorphic metric spaces and X is complete, then Y is complete.
- (b) Suppose X is a topological space with a quotient space $Y = X/\sim$ and projection map $p : X \to Y$. If $U \subseteq X$ is open, then $p(U)$ is open.
- (c) Give $\mathbb{R}$ the topology where $U \subseteq \mathbb{R}$ is open if and only if $\mathbb{R} \setminus U$ is finite or $U = \emptyset$. Then, $\mathbb{R}$ is compact with this topology.

(3) (a) Show that there is a continuous surjective map $f : S^1 \to S^2$.

(b) Show that there is no homeomorphism $f : S^1 \to S^2$.

(c) Can there be a bijective continuous map $f : S^1 \to S^2$? Prove your answer.

(4) Let $X = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$. I.e. X is the set of "grid lines" in the plane. Give X the subspace topology.

(a) Show that X is path connected.

(b) Show that X is complete if the metric d on X is the Euclidean metric.

(5) (a) Let $X = \prod_{i \in \mathbb{N}} X_i$. Show that X is Hausdorff if all X_i are Hausdorff.

(b) Show that a finite product of path-connected spaces is path-connected.

3. Fundamental Group and Covering Spaces (Mostly)

Solve 4 of the following problems. In your answers, indicate what theorems you are using.

(1) Describe the universal cover of $S^1 \vee S^2$. Prove that it is the universal cover.

(2) Show that there is no retraction $r : S^1 \times D^2 \to S^1 \times S^1$.

(3) Let $T^2 = S^1 \times S^1$ denote the 2-torus, and $\mathbb{RP}^2 = S^2 / v \sim -v$ the real projective plane.

(a) Prove that every continuous map $\mathbb{RP}^2 \to T^2$ is homotopic to a constant map.

(b) Show that there is no covering map $T^2 \to \mathbb{RP}^2$.

(4) Find counterexamples to (and use π_1 to disprove) the following statements.

(a) If X, Y, and Z are connected spaces, then $X \vee (Y \times Z)$ is homotopy equivalent to $(X \vee Y) \times Z$.

(b) If X, Y, and Z are connected spaces, then $(X \vee Y) \times Z$ is homotopy equivalent to $(X \times Z) \vee (Y \times Z)$.

(5) Suppose that X is path-connected and locally path-connected. Suppose $p : \tilde{X} \to X$ is the universal cover and $\tilde{X}$ is compact. Prove that $\pi_1(X, x)$ is finite.