Qualifying Exam: Algebra

Name:..

Please give complete arguments and use good mathematical notation. Results from the notes can of course be used. You can also use results from earlier parts of a problem later on, even if you did not answer those questions.

1. **(3+2+3+4 points)** Let $H, K \subseteq G$ be subgroups of a group G.

   (a) Give an example that shows that $HK := \{hk : h \in H, k \in K\}$ need not be a subgroup.
   (b) However, show that if $K \trianglelefteq G$, then HK is a subgroup of G.
   (c) Now assume that $H, K \trianglelefteq G$, $H \cap K = 1$. Prove that then $hk = kh$ for all $h \in H, k \in K$. Then show that the map $H \times K \to HK$, $(h, k) \mapsto hk$ defines an isomorphism.
   (d) Let G be a group of order 21. Show that either $G \cong \mathbb{Z}_3 \times \mathbb{Z}_7 (\cong \mathbb{Z}_{21})$, or G has 7 Sylow 3-subgroups.

2. **(4 points)** Let $p < q < r$ be distinct primes. Show that a group of order pqr is not simple. *Suggestion:* Try to show that for at least one of $s = p, q, r$, there is only one Sylow s-subgroup. (Perhaps assume this didn't happen, and derive a contradiction.)

3. **(3 points)** Let G be a finite group with center C. Show that either $C = G$ or $|C| \leq |G|/4$.

4. **(3+2+3 points)** Consider the group

$$G = \langle a, b | a^2 = b^3 = 1, ababab = 1 \rangle.$$

   (a) Show that G has at most 15 elements (or establish a slightly better bound if you can). *Suggestion:* First notice that every group element can be written as a word in the generators only (no inverses). Then show that every such word of length 5 or more can be rewritten as a shorter word, and finally consider words of length at most 4.
   (b) Apply Dyck's Theorem to prove that there is a homomorphism $\varphi : G \to A_4$ that sends $\varphi(a) = (12)(34)$, $\varphi(b) = (123)$.
   (c) Prove that φ from part (b) is in fact an isomorphism; in particular, G has 12 elements.

5. **(4 points)** Find all $a \in \mathbb{Z}_3$ for which the quotient ring

$$\mathbb{Z}_3[x]/(x^3 + x^2 + ax + 1)$$

   is a field.

6. **(3 points)** Let E/F be a field extension, and let $a, b \in E$ be algebraic over F with minimal polynomials of degree m and n, respectively. Show that $[F(a, b) : F] \leq mn$, and if m, n are relatively prime, then $[F(a, b) : F] = mn$.

7. **(5 points)** Let E/F be a field extension, $a \in E \setminus F$, and assume that $F(a)/F$ is Galois. Assume further that there is an automorphism $\varphi \in \mathrm{Gal}(F(a)/F)$ that maps $\varphi(a) = a^{-1}$.

   Show that then $[F(a) : F]$ is even, and $[F(a) : F] = 2[F(a + a^{-1}) : F]$. *Suggestion:* What is the order of φ? Consider associated intermediate fields.

8. **(5 points)** Find the Galois group of $f(x) = x^4 - 5x^2 + 6 \in \mathbb{Q}[x]$: describe the automorphisms. What familiar group is the Galois group isomorphic to?