January 11, 2016 **Topology Qualifying Examination** **8:30am–11:30am**

Instructions. Answer all three of Q1–Q3, and answer any three of Q4–Q7. This is six problems in 3 hours, so you should aim for an upper limit of 25–30 minutes per problem.

Q 1. Quotient Spaces.

(a) Give the definition of a *quotient map* $q : X \to Y$ between topological spaces.

(b) Let X be a topological space and $\sim$ be an equivalence relation on X. Give the definition of the *quotient space* $X/\sim$ of X by $\sim$.

(c) Let $q : X \to Y$ be a quotient map, and $f : X \to Z$ be a continuous map which is constant on the fibers (point preimages) of q. What can you conclude in this situation? (no proof necessary)

(d) State a theorem about a continuous bijection from a compact to a Hausdorff space. (no proof necessary)

(e) Let Y be the quotient space of the unit disk $D^2 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$ (with the subspace topology inherited from $\mathbb{R}^2$) obtained by identifying all points on the boundary circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Give a detailed proof that Y is homeomorphic to the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ (with the subspace topology inherited from $\mathbb{R}^3$). Give statements of results that you use in your proof.

Q 2. Let $\{X_\alpha | \alpha \in J\}$ be a non-empty indexed family of topological spaces.

(a) Define the *product topology* on $X = \prod_{\alpha \in J} X_\alpha$.

(b) Prove that the projection maps $p_\alpha : X \to X_\alpha$ are continuous.

(c) State a criterion (in terms of the projection maps p_α) for a map $f : Z \to \prod_{\alpha \in J} X_\alpha$ to be continuous.

(d) Define what it means for a topological space to be *path-connected*.

(e) Prove that if each X_α is a path-connected space, then $X = \prod_{\alpha \in J} X_\alpha$ is path-connected.

Q 3. Compactness.

(a) Define what it means for a topological space to be *compact*.

(b) Prove that a closed subspace of a compact space is compact.

(c) State the Tychonoff theorem. (no proof necessary)

(d) Let $\{X_\alpha \mid \alpha \in J\}$ be a non-empty indexed family where each $X_\alpha = \mathbb{R}$ with the standard topology, and let $X = \prod_{\alpha \in J} X_\alpha$ have the product topology.

   (i) Prove or give a counterexample: "If C is a subspace of X whose projection to each X_α is closed and bounded, then C is compact."

   (ii) Prove or give a counterexample: "If C is a closed subspace of X whose projection to each X_α is closed and bounded, then C is compact."

Q 4. Simply-connected.

(a) State (no proof necessary) the *Lebesgue number Lemma* for compact metric spaces.

(b) Define what it means for a topological space to be *simply-connected*.

(c) Suppose that a topological space $X = A \cup B$ where A and B are simply-connected open subspaces and $A \cap B$ is non-empty and path-connected. Prove that X is simply-connected.

(d) Suppose that a topological space $X = A \cup B$ where A and B are simply-connected, A is an open subspace, B is a closed subspace, and $A \cap B$ is non-empty and path-connected. Give an example, which shows that X need not be simply-connected.

Q 5. Fundamental Group and Applications.

(a) What does it mean to say that the fundamental group is a *functor* from the category of topological spaces and continuous maps to the category of groups and homomorphisms. (There are two key properties).

(b) Give the definition of a *retraction* of a topological space X onto a subspace A.

(c) Give a proof that there is no retraction from the disk $D^2 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$ to its boundary circle $\partial D^2 = S^1$. State any major results that you use in your proof.

(d) Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$. Here each space has the standard (euclidean) topology. State any results in class that you use in your proof.

Q 6. π_1 and covering spaces.

(a) Determine the *fundamental groups* of the following wedge products (one-point identifications) of spaces:

$$S^2 \vee S^2\,, \qquad P^2 \vee S^2\,, \qquad P^2 \vee P^2\,, \qquad T^2 \vee S^2\,.$$

The component spaces are as follows: $S^2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ is the 2–sphere, P^2 is the real projective plane, and $T^2 = S^1 \times S^1$ is the 2–torus.

Recall that P^2 can be defined as the quotient space of S^2 by the antipodal map equivalence relation $(x, y, z) \sim (-x, -y, -z)$.

You are free to use any known results from class notes about the fundamental groups of the 2–sphere, the projective plane and the 2–torus. State the name of the theorem that you use to compute the fundamental groups of wedge products of spaces.

(b) Draw/describe the *universal covering spaces* of each of the 4 wedge product spaces listed above.

Q 7. Covering spaces of the wedge of two circles.
Consider the wedge of two circles $X = S^1 \vee S^1$ whose fundamental group $\pi_1(X, x_0)$ is the free group $F(a, b)$ on $\{a, b\}$. In each part below, you are asked to draw covering spaces $\rho : \widetilde{X} \to X$ with the indicated properties.

(a) An infinite sheeted regular covering space.

(b) An infinite sheeted covering space which is not regular.

(c) A 3–fold covering space which is regular.

(d) A 3-fold covering space which is not regular.

(e) The image $\rho_*(\pi_1(X, x_0))$ is the normal subgroup generated by $\{a^2, b\}$.

(f) The image $\rho_*(\pi_1(X, x_0))$ is the normal subgroup generated by $\{a^3, b^2, aba^{-1}b^{-1}\}$.