August 21, 2015 **Topology Qualifying Examination** **8:30am–11:30am**

Instructions. Answer any three of the first four problems (Q1–Q4), and answer all three of Q5–Q7. This is six problems in 3 hours, so you should aim for an upper limit of 25 minutes or so per problem.

Q 1. Compact and Hausdorff spaces.

(a) Define what it means for a topological space to be **compact**.

(b) Define what it means for a topological space to be **Hausdorff**.

(c) Prove that a compact subspace of a Hausdorff space is closed.

(d) Prove that a continuous bijection f from a compact space X to a Hausdorff space Y is a homeomorphism (state the results that you use in your proof).

(e) Give an example of a continuous bijection from a non-compact space to a Hausdorff space which is not a homeomorphism.

Q 2. Connected and path-connected spaces.

(a) Define what it means for a topological space to be **connected**.

(b) Define what it means for a topological space to be **path-connected**.

(c) Prove that the continuous image of a connected space is connected.

(d) State a version of the *Intermediate Value Theorem* (from this course, or from your Calculus or Analysis courses) and show how it follows from the fact that the interval $[a, b]$ is connected.

(e) Give an example of a connected space which is not path-connected.

Q 3. One-point compactifications.

(a) Define what it means for a topological space to be **locally compact**.

(b) Describe the **one-point compactification**, $\hat{X}$, of a locally compact Hausdorff space X. That is, say what $\hat{X}$ is as a set, and then describe its topology of open sets. (You do not have to prove that the open sets form a topology).

(c) Prove that the one-point compactification $\hat{X}$ above is a Hausdorff space.

(d) Suppose that X and Y are locally compact Hausdorff spaces, and that their one-point compactifications are homeomorphic. Does it follow that X is homeomorphic to Y? (You should either sketch a proof or provide a counterexample).

Q 4. True/False.
Indicate whether each of the following is True or False. Provide a brief (one sentence or two, or provide a counterexample) reason for your answer in each case.

(a) If $X \times Y$ with the product topology is homeomorphic to $Z \times Y$ with the product topology, then X is homeomorphic to Z.

(b) Every closed subspace of a locally compact space is compact.

(c) A regular space with a countable basis is metrizable.

(d) $\mathbb{R}^\omega$ is connected in the product topology.

(e) The product of a family of metric spaces (with the product topology) is metrizable.

Q 5. Fundamental Group and Applications.

(a) What does it mean to say that the fundamental group is a **functor** from the category of topological spaces and continuous maps to the category of groups and homomorphisms. (There are two key properties).

(b) A fundamental result of the topology course is that the fundamental group of the circle $S^1\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$ is isomorphic to the infinite cyclic group $\mathbb{Z}$. Describe the isomorphism $\mathbb{Z} \to \pi_1(S^1, (1, 0))$. (You do not have to prove that it is an isomorphism).

(c) Give the definition of a **retraction** of a topological space X onto a subspace A.

(d) Give a proof that there is no retraction from the disk $D^2 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$ to its boundary circle $\partial D^2 = S^1$.

Q 6. π_1 and covering spaces of wedge products (one-point unions).

(a) Determine the **fundamental groups** of the following wedge products of spaces:

$$S^2 \vee S^2\,, \qquad A^2 \vee S^2\,, \qquad M^2 \vee S^2\,, \qquad T^2 \vee S^2\,.$$

The component spaces are as follows: $S^2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ is the 2–sphere, A^2 is the annulus (compact, with two boundary circles), M^2 is the Mobius band (compact, with one boundary circle), and $T^2 = S^1 \times S^1$ is the 2–torus.

You are free to use any known results from class notes about the fundamental groups of the 2–sphere, the annulus, the Mobius band and the 2–torus. State the name of the theorem that you use to compute the fundamental groups of wedge products of spaces.

(b) In forming $A^2 \vee S^2$ a point on the sphere is identified with a point on one of the boundary circles of the annulus. In forming $M^2 \vee S^2$ a point on the sphere is identified with a point on the boundary circle of the Mobius band. Draw/describe the **universal covering spaces** of each of the 4 wedge product spaces listed above.

Q 7. Covering spaces of the wedge of two circles.

The fundamental group of the wedge $S^1 \vee S^1$ of two circles is the free group F_2 of rank 2. For each of the four covering spaces of $S^1 \vee S^1$ shown below:

(a) Determine whether or not the corresponding subgroup of F_2 is normal.

(b) Determine whether or not the corresponding subgroup of F_2 is finite index.

(c) Compute the **automorphism group** (also known as the **deck transformation group**) of the covering space.

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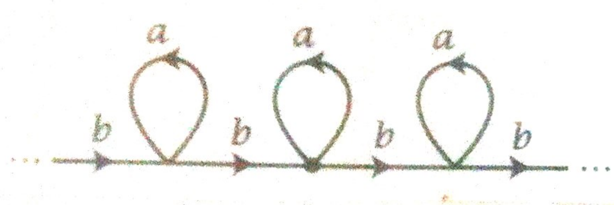

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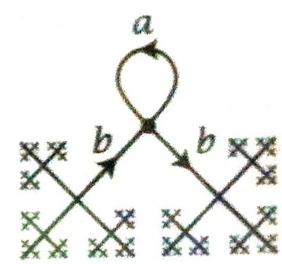

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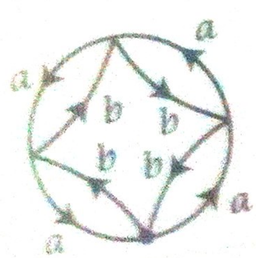

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