Qualifying examination in analysis
January 2014

1. Suppose (X, Σ, μ) is a measure space, and f is a measurable function on X with the property that for every measurable set $A \subseteq X$, if $\mu(A) > 0$ then $\int_A f \, d\mu \geq 0$. Prove that $f \geq 0$ a.e. on X.

2. *(a)* Suppose μ is a finite Borel measure on $\mathbf{R}$, $x_0 \in \mathbf{R}$, and $\{x_n\}$ is a decreasing sequence of numbers such that $\lim_{n\to\infty} x_n = x_0$. Show that

$$\lim_{n\to\infty} \mu((-\infty, x_n]) = \mu((-\infty, x_0]).$$

(b) Give an example of a finite Borel measure μ on $\mathbf{R}$, a number $x_0 \in \mathbf{R}$, and an increasing sequence of numbers $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x_0$, and

$$\lim_{n\to\infty} \mu((-\infty, x_n]) \neq \mu((-\infty, x_0]).$$

3. Suppose $\{f_n\}$ is a sequence of Lebesgue integrable functions on $[0, 1]$, and suppose f_n converges pointwise almost everywhere to a function f on $[0, 1]$. Suppose also that for every measurable subset E of $[0, 1]$ and every $n \in \mathbf{N}$, we have

$$\int_E |f_n(x)| \, dx \leq m(E).$$

Prove that f is integrable on $[0, 1]$ and

$$\lim_{n\to\infty} \int_0^1 f_n \, dx = \int_0^1 f \, dx.$$

4. Suppose f_n is a sequence of Lebesgue integrable functions on $[0, 1]$, and $|f_n(x)| \leq 1$ for all $x \in [0, 1]$ and all $n \in \mathbf{N}$. Show that if f_n converges in measure to zero, then

$$\lim_{n\to\infty} \int_0^1 |f_n(x)| \, dx = 0.$$

(Recall that we say f_n converges in measure to a function f on a set E if, for every $\eta > 0$,

$$\lim_{n\to\infty} m\{x \in E : |f_n(x) - f(x)| \geq \eta\} = 0.)$$

5. Suppose (X, Σ, μ) is a measure space, and $f \in L^p(X, \mu)$, where $1 \leq p < \infty$. Show that

$$\lim_{t\to\infty} t^p \mu\left(\{x \in X : |f(x)| > t\}\right) = 0.$$

6. Suppose f is absolutely continuous on $[0,1]$, $f(0)=0$, and $f'(x)$ is in $L^2[0,1]$. Show that

$$\lim_{x\to 0+} \frac{f(x)}{\sqrt{x}} = 0.$$

7. Suppose Λ is a bounded linear functional on $L^2(\mathbf{R})$, and $\{\phi_n\}$ is an orthonormal sequence in $L^2(\mathbf{R})$. Show that $\lim_{n\to\infty} \Lambda(\phi_n) = 0$.

8. In the measure space $(\mathbf{R}, \mathcal{M})$, where $\mathcal{M}$ is the Lebesgue σ-algebra, let λ denote Lebesgue measure and define measures μ and ν by $\mu(E) = \int_E f \; d\lambda$ and $\nu(E) = g \; d\lambda$, where

$$f(x) = \begin{cases} x+1 & (x \geq -1) \\ 0 & (x < -1) \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} x^2 & (x \geq 0) \\ 0 & (x < 0) \end{cases}.$$

Find, with proof, measures ν_1 and ν_2 such that $\nu = \nu_1 + \nu_2$, ν_1 is absolutely continuous with respect to μ, and ν_2 is singular with respect to μ.

9. For $f \in L^2(\mathbf{R})$, define the function $Tf(x)$ on $\mathbf{R}$ by

$$Tf(x) = \int_0^1 f(x+y) \; dy.$$

Show that $Tf \in L^2(\mathbf{R})$, with $\|Tf\|_2 \leq \|f\|_2$. (Here $\|\cdot\|_2$ denotes the norm in $L^2(\mathbf{R})$.)