Qualifying Examination **Real Analysis** **January 2013**

Name **Score:**

The symbols and notions are as used in the book Real Analysis by Royden 3rd ed.. If the measure is not explicitly specified measurable, integrable etc. refers to the Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^n$, respectively. You may use either the book Real Analysis by Royden 3rd ed. or Real Analysis by Fitzpatrick-Royden as reference.

Always provide a proof for your answers.

Work as many problems as you can, each problems counts for 20 pts.

The allotted time is three hours.

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1) Let (f_n) with $f_n\colon \mathbb{R} \to \mathbb{R}$ be a sequence of absolutely continuous functions, and let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers.

   i) If (f_n) is uniformly bounded (i.e.: there is $M \in \mathbb{R}$, such that $|f_n(x)| \leq M$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$), show that

   $$f\colon = \sum_{n=1}^{\infty} a_n f_n$$

   is an absolutely continuous function.

   ii) Is f is necessarily absolutely continuous in case the assumption that (f_n) is uniformly bounded is dropped.

2) Let X be the set of three elements $\{a, b, c\}$. On the collection of subsets $\mathcal{C} = \{\{\emptyset\}, \{a\}, \{a, b\}\}$ define the set function $m\colon \mathcal{C} \to [0, \infty]$ by

   $$m(\emptyset) = 0, \; m(\{a, b\}) = 1, \; m(\{a\}) = 2.$$

   i) Find an outer measure μ^* on X such that $\mu^*|_{\mathcal{C}} = m$.

   ii) Find all μ^*-measurable sets of X.

3) Let $(X, \mathcal{S}, \mu)$ be a finite measure space and let $f\colon X \to \mathbb{R}$ a measurable function.

   i) Show that $|f(x)| \leq 1$, μ-a.e., in case f^n is integrable for all n and $\int_X f^n d\mu$ is a bounded sequence.

   ii) Show: If f^n is integrable for all $n \in \mathbb{N}$, then there is a measurable set $A \subset X$ such that $f = \chi_A$ if and only if $\int_X f^n d\mu$ is constant.

Hint: Consider $\int_{A_i} f^n d\mu$ with $A_1 = \{x \in X \mid f(x) = 1\}$, $A_2 = \{x \in X \mid |f(x)| < 1\}$, $A_3 = \{x \in X \mid f(x) = -1\}$.

4) Let $f : (a, b) \to \mathbb{R}$ be a strictly increasing function, $W = f((a, b))$. Show that
   i) $f^{-1} : W \to (a, b)$ is a continuous function.
   ii) f is measurable.

5) Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be two measurable functions such that f is integrable and g is bounded. Show that

$$\lim_{t\to 0} \int_{\mathbb{R}} g(x)(f(x) - f(x - t))dx = 0\,.$$

Hint: Show first that the statement holds if f is continuous with compact support.

6) Let $(X, \mathcal{M})$ be a measurable space and let $\mu_n\colon \mathcal{M} \to [0, \infty)$ be a uniformly bounded sequence of measures., i.e.: there is a constant $c > 0$, such that $\mu_n(X) \leq c$, for all n. Show that

$$\mu = \sum_{n=1}^{\infty} \frac{\mu_n}{2^n}\,,$$

is a measure on $\mathcal{M}$ and that each μ_n is absolutely continuous with respect to μ.

7) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete measure spaces. If $A \times B$ is $\mu \times \nu$-measurable and B is ν-measurable. Does it follow that A is ν-measurable?

8) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bounded nonnegative Lebesgue-integrable function. Show that its graph $G_f = \{(x, y) \in \mathbb{R}^2 \mid f(x) = y\}$ is in measurable with respect to the two dimensional Lebesgue measure $\lambda \times \lambda$.

9) Let μ, ν and λ be s-finite measures on a measure space $(X, \mathcal{M})$, with $\nu << \mu << \lambda$. verify that we for the Randon-Nikodym derivatives we have

$$\frac{d\nu}{d\lambda} = \frac{d\mu}{d\lambda}\frac{d\mu}{d\lambda}\,, \quad \lambda\text{-almost everywhere.}$$

10) Let μ be a Borel measure on $[0, 1]$ which is absolutely continuous with respect to the Lebesgue measure. Show that its Radon-Nikodym derivative is the derivative of its cumulative distribution function.