Qualifying Examination **Real Analysis** **May 20, 2009**

Name: ________________________________

Problem 1. [0+2+3+3+2+5 points]
Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, right-continuous function defined by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \ , \\ 3 + x & \text{if } x \in [0, 2) \ , \\ 5 & \text{if } 2 \leq x \ , \end{cases}$$

and let μ_F be the Borel measure associated with F. Let m stand for the Lebesgue measure on $\mathbb{R}$.

(a) Sketch the function F.

(b) Give a one-line argument showing that μ_F is *not* absolutely continuous with respect to m.

(c) Write down the Lebesgue-Radon-Nikodym representation, $\mathrm{d}\mu_F = f\,\mathrm{d}m + \mathrm{d}\lambda$, of μ_F with respect to m. What is f? Why is the question "What is $f(1)$?" meaningless?

(d) Identify clearly the discrete, the absolutely continuous, and the singularly continuous parts of the measure μ_F.

(e) Compute $\displaystyle\int_5^7 \cos(x^2)\,\mathrm{d}\mu_F(x)$.

(f) Now think of the function F as a distribution (denoted by the same letter), i.e., let $F \in \mathcal{D}'(\mathbb{R})$ be defined by $\langle F, \phi \rangle = \int_{\mathbb{R}} F(x)\,\phi(x)\,\mathrm{d}m(x)$. What does the general theory say about the derivative, $F' \in \mathcal{D}'(\mathbb{R})$, of the distribution F? Write down $\langle F', \phi \rangle$ in terms of the values and integrals of the test function $\phi \in \mathcal{D}(\mathbb{R})$.

Problem 2. [5 points]

Let ν be a signed measure. Prove that E is ν-null if and only if $|\nu|(E) = 0$.

Hint: Use the Hahn decomposition for ν.

Problem 3. [4+4 points]

(a) Prove that the set of real-valued Borel measurable functions on $\mathbb{R}$ is closed under functional composition. In other words, show that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are Borel measurable, then $f \circ g$ is Borel measurable.

(b) Give an example showing that the set of all real-valued functions on $\mathbb{R}$ that are L^1 is *not* closed under functional composition. (I.e., find a pair of L^1 functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g$ is not in L^1.)

Problem 4. [3+3+3+3+3 points]
Let $X = \{a, b, c\}$, $\mathcal{M} = \mathscr{P}(X)$ (i.e., all subsets of X are in the σ-algebra $\mathcal{M}$), and the measure $\mu : \mathcal{M} \to [0, \infty]$ be defined by

$$\mu(\{a\}) = 0 \; , \qquad \mu(\{b\}) = 1 \; , \qquad \mu(\{c\}) = \infty \; .$$

Consider the set of all real-valued functions on X. Define the operations "addition" and "multiplication by a scalar" as usual: if $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$, $\alpha \in \mathbb{R}$, then $(f + g)(j) := f(j) + g(j)$, $(\alpha f)(j) := \alpha f(j)$ for $j \in \{a, b, c\}$. Clearly, the set of real-valued functions on X forms a real vector space.

(a) **[3 points]** What is the most general form of a real-valued function on X? What is the dimension of the real vector space of such functions?

(b) **[3 points]** What is the most general form of all real-valued functions on X satisfying $\int |f| \, \mathrm{d}\mu < \infty$?

(c) **[3 points]** Prove that the functions considered in part (b) form a real vector space.

(d) **[3 points]** What is the dimension of the vector space from part (c)? Explain briefly.

(e) **[3 points]** What is the dimension of the space $L^1(X, \mu)$? Explain your reasoning.

Problem 5. [4+4 points]

(a) Let X and Y be arbitrary sets, and $f : X \to Y$ be an arbitrary function. Let J be a countable set, and $\{E_j\}_{j \in J}$ be an arbitrary family of subsets of Y. Prove that $f^{-1}\left(\bigcup_{j \in J} E_j\right) = \bigcup_{j \in J} f^{-1}(E_j)$.

(b) Let $(X, \mathcal{M}, \mu)$ be a measure space, and $g : X \to \overline{\mathbb{R}}$ be a function from the set X to the extended real line such that $g^{-1}((r, \infty]) \in \mathcal{M}$ for each rational number $r \in \mathbb{Q}$. Use your result from part (a) to prove that the function g is measurable.

Problem 6. **[4+4 points]**

Let ν be a signed measure on the measurable space $(X, \mathcal{M})$, and ν^+, ν^-, and $|\nu|$ be its positive, negative, and total variations, respectively.

(a) Prove that $L^1(\nu) = L^1(|\nu|)$.

(b) Show that, for any $f \in L^1(\nu)$, $\left| \int f \, \mathrm{d}\nu \right| \leq \int |f| \, \mathrm{d}|\nu|$.

Problem 7. [4+4+4+4 points]

(a) Prove that the only solution of the equation $x(x-1)^2\,u(x) = 0$ in the space of continuous functions, $u \in C(\mathbb{R})$, is the function $u(x) = 0$ for all $x \in \mathbb{R}$.

(b) For $a \in \mathbb{R}$, let $\delta_a \in \mathcal{D}'(\mathbb{R})$ be the distribution defined as $\langle \delta_a, \phi \rangle = \phi(a)$ for $\phi \in \mathcal{D}(\mathbb{R})$. Recall that in class and in tests we proved the relations $\psi\,\delta_a = \psi(a)\,\delta_a$, $\psi\,\delta'_a = \psi(a)\,\delta'_a - \psi'(a)\,\delta_a$, and $\psi\,\delta''_a = \psi(a)\,\delta''_a - 2\psi'(a)\,\delta'_a + \psi''(a)\,\delta_a$, for any $\psi \in C^\infty(\mathbb{R})$. Show that the equation $x(x-1)^2\,u(x) = 0$ has a non-zero solution in the space of distributions $\mathcal{D}'(\mathbb{R})$ – namely, the distribution $C_1\delta_0 + C_2\delta_1 + C_3\delta'_1$ satisfies the equation.

(c) Derive an expression for $\psi\delta_a'''$ in terms of the values of the function $\psi \in C^\infty(\mathbb{R})$ and its derivatives at a, and derivatives of δ_a.

(d) Show by direct substitution that δ_a''' is not a solution of $x(x-1)^2\,u(x) = 0$.

Problem 8. [3+2+2+5 points]

(a) Let $f \in L^1(\mathbb{R})$. Prove that $\lim_{R\to\infty} \int_R^\infty |f(x)|\,\mathrm{d}x = 0$. Specify which theorems you use in your proof.

(b) Write the precise mathematical definition of the statement

"the function $f : \mathbb{R} \to \mathbb{R}$ does not tend to 0 as its argument tends to ∞".

(c) Write the precise mathematical definition of the statement

"the function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous".

(d) Use parts (a)-(c) to show that, if $f \in L^1(\mathbb{R})$ and f is uniformly continuous, then $\lim_{x\to\infty} f(x) = 0$.

Problem 9. [6 points]

Let $f \in L^p(\mathbb{R})$. If there exists $h \in L^p(\mathbb{R})$ satisfying

$$\lim_{y \to 0} \left\| \frac{\tau_{-y} f - f}{y} - h \right\|_p = 0 \ ,$$

we call h the (strong) L^p-derivative of f. Let p and q be conjugate exponents, $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$, and the strong L^p-derivative of f exists. Prove that the function $(f * g)$ is differentiable in the *ordinary* sense and $(f * g)' = (f') * g$ (where f' is the strong L^p-derivative of f).

Problem 10. [3+3+3+4+4 points]
Let $(X, \mathcal{M}, \mu)$ be a probability space (i.e., a measure space with $\mu(X) = 1$), let $f : X \to \mathbb{R}$ be in $L^1(\mu)$, and $\mathcal{A} \subset \mathcal{M}$ be a σ-subalgebra of $\mathcal{M}$.

(a) Prove that $\nu : A \mapsto \int_A f(x)\,\mathrm{d}\mu(x)$ (for $A \in \mathcal{A}$) defines a signed measure on $(X, \mathcal{A})$.

(b) Show that the signed measure ν defined in (a) is absolutely continuous with respect to $\mu|_{\mathcal{A}}$.

(c) Use the Lebesgue-Radon-Nikodym Theorem to show the existence of a $\mathcal{A}$-measurable function $g : X \to \mathbb{R}$ satisfying

$$\int_A g(x)\,\mathrm{d}\mu(x) = \int_A f(x)\,\mathrm{d}\mu(x) \qquad \text{for all } A \in \mathcal{A}\ .$$

(d) Let $X = [0,1]$ and μ be the Lebesgue measure on X. Let $\mathcal{A}$ be the σ-algebra generated by the sets $[0, \frac{1}{2}]$ and $[\frac{1}{3}, 1]$. Write down all the sets in $\mathcal{A}$.

Hint: $\mathcal{A}$ consists of exactly 8 subsets of $[0,1]$.

(e) Let $f : [0,1] \to \mathbb{R} : x \mapsto x^2$, and $g : [0,1] \to \mathbb{R}$ be as in (c). Find the explicit expression for g.

Hint: The function g is a sum of three indicator functions.

Problem 11. [4+4 points]

(a) Show that the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto \mathrm{e}^x$ does *not* define a tempered distribution.

(b) Prove that, if $h : \mathbb{R} \to \mathbb{R}$ is a bounded function in $C^1(\mathbb{R})$, then $g : \mathbb{R} \to \mathbb{R} : x \mapsto \mathrm{e}^x h'(\mathrm{e}^x)$ defines a tempered distribution.

Hint: Use integration by parts to show that $\langle g, \phi \rangle$ is bounded from above by a constant times $\|\phi\|_{(2,1)}$, where $\|\phi\|_{N,\alpha} = \sup_{x\in\mathbb{R}}(1 + |x|)^N \, |\partial^\alpha \phi(x)|$.