Topology Qualifying Review Exam

May 14, 2007

• Do all the problems <u>in the right order</u>.

1. Let Y be a Hausdorff space; and $f, g : X \longrightarrow Y$ be continuous maps. Suppose $f = g$ on a subset A of X which is dense in X. Prove that $f = g$ on X.

2. Let $\{x_n : \ n = 1, 2, 3, \cdots\}$ be a sequence of points in a topological space X, converging to x_0. Prove that the set $K = \{x_n : \ n = 0, 1, 2, 3, \cdots\}$ is compact.

3. For each $\alpha \in J$, let X_α be a topological space with topology $\mathfrak{T}_\alpha$. Let $X = \prod_{\alpha \in J} X_\alpha$ be given the product topology, and let $\pi_\alpha : X \to X_\alpha$ denote the projection map. Prove that a function $f : Y \to X$ is continuous if each $\pi_\alpha \circ f$ is continuous.

4. Let C be the set of all continuous real-valued functions on $[0, 1]$. For each $f \in C$ and $\epsilon > 0$, define

$$M(f, \epsilon) = \{g \in C : \int_0^1 |f - g| < \epsilon\}$$
$$U(f, \epsilon) = \{g \in C : \sup_{x \in [0,1]} |f(x) - g(x)| < \epsilon\}.$$

(1) Prove that $\mathfrak{M} = \{M(f, \epsilon) : f \in C, \epsilon > 0\}$ forms a basis for a topology.
(2) Compare the two topologies generated by $\mathfrak{M}$ and $\mathfrak{U} = \{U(f, \epsilon) : f \in C, \epsilon > 0\}$,

5. Let $\mathfrak{A} = \{A_\alpha : \alpha \in J\}$ be a locally-finite collection of closed covering of a space X. Let $f : X \to Y$ be a function, and suppose $f|_{A_\alpha}$ (restriction of f to A_α) is continuous for each $\alpha \in J$. Prove that f is continuous.

6. Let D be the unit disk in $\mathbb{R}^2$ with the subspace topology. We identify all the boundary points of D (to one point p), and give the quotient topology to the resulting quotient set Q. Prove that Q is homeomorphic to the sphere S^2. Pay special attention to the local neighborhood system of Q at the point p.

7. Let X be a locally compact Hausdorff space. For a space Y, the set of all continuous maps from X to Y is denoted by $C(X, Y)$. It has the compact-open topology. Prove the map $e : X \times C(X, Y) \to Y$ defined by

$$e(x, f) = f(x)$$

is continuous.

8. Given a path f in a space X from x_0 to x_1, let $\overline{f}$ be the path in X defined by $\overline{f}(s) = f(1 - s)$. Prove that $f * \overline{f}$ is homotopic to the constant path e_{x_0} at x_0.

9. (a) State Seifert–van Kampen theorem (classical version, to be applied to the next question).
(b) Use (a) to calculate the fundamental group of the following space: A space A is a torus with an open disk D removed. Let $f : \partial B \to \partial A$ be a map from the boundary of a 2-ball B to the boundary of A winding twice (that is, double covering map from a circle to a circle). Let X be the space joining the 2-cell B by the map f. What is the fundamental group of X?

10. Let $p : (E, e_0) \to (B, b_0)$ be a covering map. Let $f : (Y, y_0) \to (B, b_0)$ be a continuous map. Suppose Y is path connected and locally path connected. If $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0))$, the f can be lifted to a map $\widetilde{f} : (Y, y_0) \to (E, e_0)$.

11. Let $p : X \to B$ be a regular covering map; let G be its group of covering transformations (so that the action of G is properly discontinuous). Let $\pi : X \to X/G$ be the projection map. Show that there is a homeomorphism $k : X/G \to B$ such that $k \circ \pi = p$.

$$\begin{array}{ccc} X & \xrightarrow{=} & X \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X/G & \xrightarrow{k} & B \end{array}$$

12. The group $E(2) = \mathbb{R}^2 \rtimes \mathrm{O}(2)$ (where $\mathrm{O}(2)$ is the orthogonal group) is $\mathbb{R}^2 \times \mathrm{O}(2)$ as sets, but the group operation is given by $(a, A) \cdot (b, B) = (a + Ab, AB)$.
(a) $E(2)$ acts on the space $\mathbb{R}^2$ by $(a, A) \cdot x = a + Ax$ for $x \in \mathbb{R}^2$. Show this actually defines an action.
(b) Let π be the subgroup of $E(2)$ generated by the 3 elements $(e_1, I), (e_2, I)$ and (a, A), where

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \text{ and } A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Show π contains a subgroup $\mathbb{Z}^2$ of index 2.
(c) Is the action of π on $\mathbb{R}^2$ free?
(d) Identify the orbit space $\mathbb{R}^2/\pi$. Explain how you derive the conclusion.