

## Topology Qualifying Examination

August, 2005

*Instructions:* Do as many problems as you can. Try to give complete arguments, but do not spend excessive time verifying obvious details, especially when giving examples. Apply major theorems when possible.

All metric spaces are assumed to have the metric topology. All products of topological spaces are assumed to have the product topology. The spaces  $I = [0, 1]$ ,  $\mathbb{R}$ ,  $\mathbb{R}^n$ , and  $S^n$  are assumed to have their standard metrics and topologies, unless otherwise stated. In any problem involving fundamental groups or covering maps, it is assumed that all spaces involved are connected, locally path-connected, semilocally simply-connected, and Hausdorff.

1. Let  $X$  and  $Y$  be nonempty topological spaces. Prove that  $X \times Y$  is Hausdorff if and only if both  $X$  and  $Y$  are Hausdorff.
2. Prove that the lower limit topology on  $\mathbb{R}$  does not have a countable basis.
3. Prove that any continuous map from the real projective plane  $\mathbb{P}^2$  to  $S^1$  is homotopic to a constant map.
4. Recall that a *knot* in  $\mathbb{R}^3$  is the image  $K$  of an imbedding  $j: S^1 \rightarrow \mathbb{R}^3$ .
  - (a) Prove that no knot can be a retract of  $\mathbb{R}^3$ .
  - (b) Prove that if  $K_1$  and  $K_2$  are knots, then there exists a continuous map  $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  that carries  $K_1$  homeomorphically to  $K_2$ .
5. Give examples of each of the following.
  - (a) A metrizable space that is not locally compact.
  - (b) A metrizable space in which every connected component is a single point and every  $\epsilon$ -ball is uncountable.
  - (c) A sequence of continuous functions  $f: I \rightarrow I$  that converges pointwise but not uniformly to a continuous function  $g: I \rightarrow I$ .
6. Recall that if  $p: F \rightarrow G$  is a  $k$ -fold covering map between compact surfaces, then  $\chi(F) = k\chi(G)$ . Let  $G = T \# T \# D^2 \# D^2$ , the orientable surface of genus 3 with two boundary circles. Show that if  $F$  is a compact orientable surface that is a covering space of  $G$ , then  $F$  has an even number of boundary circles.
7. Let  $\mathcal{C}(\mathbb{R}, \mathbb{R})$  be the space of continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$ , with the compact-open topology. For each  $n$ , let  $f_n: \mathbb{R} \rightarrow \mathbb{R}$  be the constant function defined by  $f_n(x) = n$  for all  $x$ . Prove that the sequence  $\{f_n\}_{n=1}^\infty$  in  $\mathcal{C}(\mathbb{R}, \mathbb{R})$  has no convergent subsequence.
8. Let  $H = \prod_{k=1}^\infty [0, 1]$ . For  $n \in \{2, 3, \dots\}$  let  $H_n = \prod_{k=1}^\infty [0, 1 - 1/n]$ , and let  $H_\infty = \cup_{n=2}^\infty H_n$ .
  - (a) Prove that for each  $n \in \{2, 3, \dots\}$ , there exists a retraction from  $H$  to  $H_n$ .
  - (b) Prove that there does not exist a retraction from  $H$  to  $H_\infty$ .
9. Let  $n \geq 1$ , and let  $h_N: S^n - \{(0, \dots, 0, 1)\} \rightarrow \mathbb{R}^n$  and  $h_S: S^n - \{(0, \dots, 0, -1)\} \rightarrow \mathbb{R}^n$  be the stereographic projection homeomorphisms, projecting from the north pole and south pole respectively. Use  $h_N$  and  $h_S$  to prove that  $S^n$  is path-connected.
10. Let  $X \subset \mathbb{R}^3$  be the union of  $S^2$  and one of its diameters. Draw the universal cover of  $X$ . What well-known group is  $\pi_1(X)$ ?
11. Recall that a manifold is *closed* if it is compact and has empty boundary. Prove that a closed connected  $n$ -manifold cannot be imbedded in a connected  $n$ -manifold that has nonempty boundary.
12. Let  $C \subset \mathbb{R}^2$  be the comb space which is the union of  $I \times \{0\}$ ,  $\{0\} \times I$ , and  $\{1/n\} \times I$  for  $n \in \{1, 2, 3, \dots\}$ . Prove that there does not exist any continuous surjective map from  $I$  to  $C$ .