Topology Qualifying Examination
May, 2005

Instructions: Try to give complete arguments, but do not spend excessive time verifying obvious details, especially when giving examples. Apply major theorems when possible.

All metric spaces are assumed to have the metric topology. All products of topological spaces are assumed to have the product topology. The spaces $I = [0,1]$, $\mathbb{R}$ and $\mathbb{R}^n$ are assumed to have their standard metrics and topologies. In any problem involving fundamental groups or covering maps, it is assumed that all spaces involved are connected, locally path-connected, semilocally simply-connected, and Hausdorff.

Part A. Try to do all of these problems.

1. Let $f\colon X \to Y$ be a function between topological spaces. Prove that f is continuous if and only if for every x and every open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(U) \subseteq V$.

2. Prove that a subspace A of a space X is dense if and only if its complement $X - A$ has empty interior.

3. Give examples of each of the following, without verification.

(a) A connected space that is not path-connected.
(b) A path-connected space that is not locally connected.
(c) A metrizable space that is not locally compact.
(d) A sequence of functions from I to I which converges pointwise but not uniformly.
(e) A connected Hausdorff space which is not second countable.

4. Let $p\colon E \to B$ be a covering map. Prove that if E is compact, then $p^{-1}(b)$ is finite for every $b \in B$.

5. A subset U of $\mathbb{R}$ is called *scattered* if $U = \cup_{i=1}^{\infty} U_i$ where each U_i is a (nonempty) open interval, and $U_i \cap U_j = \emptyset$ if $i \neq j$. Let $\mathcal{B}$ be the collection of all scattered subsets of $\mathbb{R}$. Prove that $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}$.

6. Find all the compact connected surfaces which have at least three boundary circles and have Euler characteristic -5.

7. Let M be a connected n-manifold whose universal covering space is $\mathbb{R}^n$. Prove that for any $k \geq 2$, any continuous map from S^k to M is homotopic to a constant map.

8. Let X be a topological space and suppose that $X = \cup_{i=1}^{\infty} X_i$ with $X_i \subseteq X_{i+1}$ for each i.

(a) Prove that if each X_i is path-connected, then X is path-connected.
(b) Give an example where X is path-connected although no X_i is path-connected.
(c) Prove that if each X_i is an open subset of X, and $f\colon C \to X$ is a continuous map with C compact, then $f(C) \subseteq X_N$ for some N.
(d) Prove that if each X_i is simply-connected and is an open subset of X, then X is simply-connected.

9. Let $\mathcal{C}(\mathbb{R}, \mathbb{R})$ be the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, with the compact-open topology. Prove that the subspace $\mathcal{B}(\mathbb{R}, \mathbb{R})$ consisting of the bounded functions is a dense subset of $\mathcal{C}(\mathbb{R}, \mathbb{R})$.

10. Let $x_0 \in A \subseteq X$, and let $i\colon A \to X$ be the inclusion. Prove that if there exists a retraction $r\colon X \to A$, then $i_{\#}\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is injective.

Part B. Do as many of these problems as you can.

1. Let $j_1, j_2 \colon [0,1] \to \mathbb{R}^n$ be imbeddings. Suppose that $\{j_1(1)\} = \{j_2(0)\} = j_1(I) \cap j_2(I)$. Prove that there exists a continuous map $f \colon \mathbb{R}^n - \{j_1(1)\} \to \mathbb{R}$ such that $f \circ j_1(x) = x$ for all $x \in [0,1)$ and $f \circ j_2(x) = x$ for all $x \in (0,1]$.

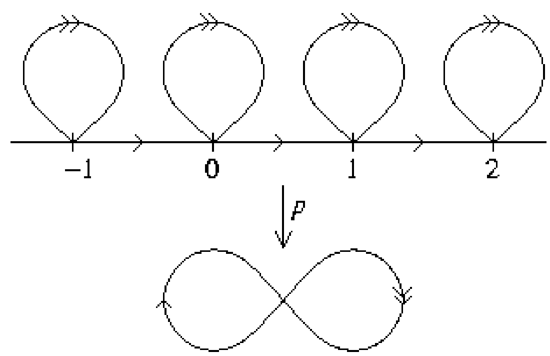
2. A continuous map $f \colon X \to \mathbb{R}$ is called *proper* if $f^{-1}([-n,n])$ is compact for each positive integer n. For a proper f, define the *two-point compactification* of X *relative to* f to be the union $X_f = X \cup \{x_+, x_-\}$, where x_+ and x_- are two distinct points not in X, with topology defined as follows: U is open in X_f when $U \cap X$ is open in U and moreover

(1) if $x_+ \in U$ then U contains $f^{-1}((r,\infty))$ for some $r \in \mathbb{R}$, and
(2) if $x_- \in U$ then U contains $f^{-1}((-\infty,r))$ for some $r \in \mathbb{R}$.

You do not need to verify that this defines a topology on X.

(a) Prove that if X is Hausdorff, then X_f is Hausdorff.
(b) Prove that X_f is compact.
(c) Give an example of a space X and two proper maps $f, g \colon X \to \mathbb{R}$ such that X_f is not homeomorphic to X_g.

3. A covering space E of $B = S^1 \vee S^1$ is shown at the right. It continues in the same pattern to the left and to the right forever. As usual, the single arrows cover the circle which corresponds to an element $a \in \pi_1(B, b_0)$, and the double arrows cover the other circle, which corresponds to an element $b \in \pi_1(B, b_0)$. Write G for $\pi_1(B, b_0)$ and H for $p_\#(\pi_1(E, e_0))$. Regard the union of the single arrow edges as $\mathbb{R}$, with the double arrow edges attached at the integers, and let the basepoint be $e_0 = 0$.



(a) Tell how one knows that the elements $\ldots, a^{-1}ba, b, aba^{-1}, a^2ba^{-2}, \ldots$ are in H.

(b) More generally, explain why an element $a_1^{m_1} b_1^{n_1} a_2^{m_1} b_2^{n_2} \cdots a_k^{m_k} b_k^{n_k}$ of G is in H if and only if $m_1 + m_2 + \cdots + m_k = 0$.

(c) Explain why the right cosets of H are Ha^n, $n \in \mathbb{Z}$.

(d) What is the group of covering transformations for this covering space? Why?

4. Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous map such that $f^{-1}([-n,n])$ is compact for each positive integer n. Prove that f achieves either a minimum value or a maximum value. (Remark: As far as I know, problem B2 above is not useful here.) Hint: One possible solution makes use of the fact that $\mathbb{R}^2 - D_N^2$ is connected, where D_N^2 is the disk of radius N centered at the origin.

5. Let $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ be a nonempty collection of topological spaces, with each $X_\alpha = \mathbb{R}$, and for each α let $\pi_\alpha \colon \prod_{\alpha \in \mathcal{A}} X_\alpha \to X_\alpha$ denote the projection map from the product to the factor X_α.

(a) Prove or give a counterexample: If C is a subset of $\prod_{\alpha \in \mathcal{A}} X_\alpha$ whose projection to each X_α is a closed bounded subset, then C is compact.

(b) Prove or give a counterexample: If C is a closed subset of $\prod_{\alpha \in \mathcal{A}} X_\alpha$ whose projection to each X_α is a closed bounded subset, then C is compact.